Section 15.8

The Method of Lagrange Multipliers

Our goal in this section is to optimize a function $f(x_1, x_2, \ldots, x_n)$ with a given constraint $g(x_1, x_2, \ldots, x_n) = 0$. *Cf.* the problem in Section 15.7 of finding absolute extrema including the analysis of the boundary of a closed region.

Theorem 15.15: Parallel Gradients
Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x, y) = 0$. Assume f has a local extreme value on C at a point $P(a, b)$. The $\nabla f(a, b)$ is orthogonal to the line tangent to C at P. If $\nabla g(a, b) \neq \vec{0}$, then there is a real number λ such that

$$\nabla f(a, b) = \lambda \nabla g(a, b).$$

Method of Lagrange Multipliers
Let the objective function f and the constraint function g be differentiable on a region of $\mathbb{R}^2$ with $\nabla g(x, y) \neq \vec{0}$ on the curve $g(x, y) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y) = 0$:

1. Find the values x, y, and λ that satisfy the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad \text{and} \qquad g(x, y) = 0.$$

2. Evaluate f at the values (x, y) found in Step 1 and at the endpoints of the constraint curve (if they exist). The largest such value is the absolute maximum; the smallest the absolute minimum of f subject to the constraint.

Example 15.8.1. Find the absolute extrema of $f(xly) = e^{xy}$ subject to the constraint $x^2 + xy + y^2 = 9$.

Example 15.8.2. Find the points on the cone $z^2 = x^2 + y^2$ closest to the point $P(1, 2, 0)$.